axioms

MDPI

Article

Pre-Emptive-Weights Goal-Programming for a Multi-Attribute Decision-Making Problem with Positive Correlation among Finite Criteria

Juin-Han Chen [1] and Hui-Chin Tang [2]

[1] Department of Industrial Engineering and Management, Cheng Shiu University, Kaohsiung City 83347, Taiwan
[2] Department of Industrial Engineering and Management, National Kaohsiung University of Science and Technology, Kaohsiungu City 80778, Taiwan

Abstract: This paper analyzes the various properties of the positively correlated weights related to the subset of finite criteria in a multi-attribute decision-making problem. Given a finite number of criteria, the exact constraints of the positively correlated weights related to the subset of criteria are presented. Introducing the non-Archimedean number, the associated bounded polyhedral-set is shown. The number of the extreme points in the bounded polyhedral-set will increase as the number of criteria increase. Applying the proposed efficient extreme-point method, the pre-emptive-weights-goal-programming optimal solution is shown. These theoretical global-maximum values of the positively correlated weights related to the subset of finite criteria are useful for practical applications.

Keywords: multi-attribute decision-making; correlation; extreme points; goal-programming

MSC: 90B50; 90C29; 90C49; 62H20

Citation: Chen, J.-H.; Tang, H.-C. Pre-Emptive-Weights Goal-Programming for a Multi-Attribute Decision-Making Problem with Positive Correlation among Finite Criteria. *Axioms* **2023**, *12*, 20. https://doi.org/10.3390/axioms12010020

Academic Editors: Radko Mesiar

Received: 11 November 2022
Revised: 11 December 2022
Accepted: 22 December 2022
Published: 25 December 2022



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Consider a finite set of alternatives $A = \{a, b, c, \dots\}$ and a finite set of attributes/criteria $N = \{1, 2, \dots, n\}$ in a multiattribute decision-making (MADM) problem [1–6]. MADM problems are wide spread in real-life situations. An MADM problem is to rank the alternatives or select one of the best solutions from feasible alternatives, which are assessed on multiple attributes. For an MADM problem, two issues should be determined by the decision maker. One is the ratings of alternatives on attributes for a decision matrix. The other is the weights of attributes for an aggregation operator. Let x_i^a be ratings of alternatives $a \in A$ on attributes $i \in N$. All these values are concisely expressed in the vector format as $x^a = (x_1^a, x_2^a, \dots, x_n^a) \in \mathcal{R}^n$, which is usually denoted by $x = (x_1, x_2, \dots, x_n)$ for short. A profile $x = (x_1, x_2, \dots, x_n)$ is referred to a score of an alternative $a \in A$ related to criteria N. Assume that weights of attributes $j \in N$ are w_j, which satisfy the normalization conditions: $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

From the profile of any alternative, $a \in A$, and the weights of attributes, one can compute a global score by means of an aggregation operator $M: \mathcal{R}^n \to \mathcal{R}$. The most often used aggregation operators are the weighted arithmetic means [2]

$$M(x) = \sum_{i=1}^{n} w_i x_i. \quad (1)$$

Once the global scores are computed, we adopt the value of $M(x)$ to yields the ranking of the alternatives or select the best alternative.

There are two ways of interaction among criteria. One is that the criteria are independent and exhaustive. The other one is the complex interaction between criteria [7]. The weighted arithmetic mean $M(x)$ can be used only in the presence of independent criteria. In many practical applications, the decision criteria present some interaction. It is useful to substitute the weight vector $(w_1, w_2, \dots, w_n)$ a nonadditive set function on N. For this purpose, the concept of fuzzy measure [8] has been introduced.

A fuzzy measure on N is a set function v: $2^N \to [0,1]$ with

(i) monotonicity condition: $v(S) \geq v(T)$ for $S \supseteq T,\ S, T \subseteq N$.
(ii) boundary conditions: $v(\emptyset) = 0$ and $v(N) = 1$.

Throughout this paper, the set of all fuzzy measures on N is denoted by $\mathfrak{F}_N$.

A generalized aggregation operator, which generalizes the weighted arithmetic mean $M(x)$, is the discrete Choquet integral $\mathcal{C}_v(x)$: $\mathcal{R}^n \to \mathcal{R}$ [9,10]. The Choquet integral of $x = (x_1, x_2, \dots, x_n) \in \mathcal{R}^n$ with respect to $v \in \mathfrak{F}_N$ can be expressed in the following form:

$$\mathcal{C}_v(x) = \sum_{i=1}^{n} x_{(i)}\left[v\left(A_{(i)}\right) - v\left(A_{(i+1)}\right)\right] \quad (2)$$

where (.) is a permutation of N, such that $x_{(1)} \leq x_{(2)} \dots \leq x_{(n)}$ and $A_{(i)} = \{(i), (i+1), \dots, (n)\}$, $A_{(n+1)} = \emptyset$. The Choquet integral explicitly captures the importance of not only individual criteria, but of their subsets, as well as various interactions between the criteria. If the fuzzy measure $v \in \mathfrak{F}_N$ is additive, that is, $v(S \cup T) = v(S) + v(T)$ for $S \cap T = \emptyset,\ S, T \subseteq N$, then the Choquet integral is reduced to the weighted arithmetic mean

$$\begin{aligned}\mathcal{C}_v(x) &= \sum_{i=1}^{n} x_{(i)}\left[v\left(A_{(i)}\right) - v\left(A_{(i+1)}\right)\right] \\ &= \textstyle\sum_{i=1}^{n} x_{(i)}\left[v\left((i)\right) + v\left((i+1)\right) + \cdots + v\left((n)\right) - \left[v\left((i+1)\right) + \cdots + v\left((n)\right)\right]\right] \\ &= \textstyle\sum_{i=1}^{n} x_{(i)} v((i)).\end{aligned}$$

Similarly, if the fuzzy measure $v(S)$ is cardinality-based, the Choquet integral collapses into the ordered weighted averaging. The other particular case of Choquet integrals is the 2-additive measure [11], where only interactions I_{ij} between two criteria *i* and *j* are taken into account, and higher order interactions are ignored. This type of Choquet integral [11] is defined by

$$\mathcal{C}_v(x) = \textstyle\sum_{i=1}^{n}\left(v_i - \frac{1}{2}\sum_{j \neq i}\left|I_{ij}\right|\right)x_i + \sum_{I_{ij}>0} I_{ij}\min\{x_i, x_j\} + \sum_{I_{ij}<0} I_{ij}\max\{x_i, x_j\}$$

where v_i represents the importance of criterion *i* relative to all the others with $v_i \geq \frac{1}{2}\sum_{j \neq i}\left|I_{ij}\right|$ and $\sum_{i=1}^{n} v_i = 1$, and I_{ij} represents the interactions between pairs of the criteria *i* and *j*, with values lying in the interval $[-1, 1]$. $I_{ij} = 1$ means that two criteria *i* and *j* are complementary. $I_{ij} = -1$ indicates redundancy, and $I_{ij} = 0$ implies non interaction among criteria. Büyüközkan et al. [12,13] used the 2-additive Choquet integral to determine the correlations between two criteria for the fourth-party-logistics operating-model selection, and the software-development risk assessment, respectively.

In the literature, there are some induced aggregation operators based on the Choquet integral, which take as their argument pairs in which one component called the order-inducing variable is used to induce an ordering over the second component, called the argument variable, and they are then aggregated. For the inuitionistic-fuzzy sets, Xu [14] and Tan and Chen [15] developed the intuitionistic-fuzzy-Choquet ordered-averaging operator and induced-Choquet ordered-averaging operator based on a Choquet integral,

respectively. Xu and Xia [16] proposed the induced-generalized intuitionistic-fuzzy-Choquet integral operators and the induced-generalized intuitionistic-fuzzy-Dempster–Shafer operators. Baydaş and Elma [17] and Mukhametzyanov [18] proposed new approaches to deal with criteria weights in the decision-making field.

For the interaction among criteria, there are three types of dependence between criteria in the literature: correlation [11], substitutivity/complementarity [19], and preferential dependence [2,20,21]. Correlation is probably the best known and the most easy-to-understand type of dependence. Two criteria $i, j \in N$ are positively correlated if one can observe a positive correlation between the partial scores related to *i* and those related to *j*. If one is good for criterion *i* , it is also good for criterion *j*, and vice versa. The global evaluation will be overestimated (and underestimated, respectively), for one good (or bad, respectively), for criterion *i* and/or criterion *j*. This paper focuses on the positively correlated weights related to the subset of criteria *N*. The results of the negative correlation for each coalition of criteria can be derived similarly.

To my knowledge, the behaviors of the positively correlated weights $v(S)$ related to the subset $S \subseteq N$ of the criteria have not been yet proposed for an MADM problem with finite criteria. We present the associated feasible region. By introducing the non-Archimedean number, the bounded polyhedral set is presented. Maximizing the positively correlated weights, $v(S)$, for $S \subseteq N$, the multicriteria programming problem is proposed. Applying the pre-emptive-weights goal-programming, the pre-emptive-weights goal-programming optimal solutions are shown. We have the theoretical global-maximum values of the positively correlated weights, related to the subset of finite criteria.

The organization of this paper is as follows. Section 2 briefly reviews the three types of dependence. We present the exact constraints of the weights $v(S)$, $S \subseteq N$, of the coalition of criteria for an MADM problem with positive correlation among finite criteria. The multicriteria programming problem of the positive correlation is proposed in Section 3. We present the pre-emptive-weights goal-programming optimal solutions for an MADM problem with positive correlation among finite criteria. Finally, some concluding remarks and future research are presented.

2. Polyhedral Set of an MADM Problem with Positive Correlation among Finite Criteria

We firstly examine the basic notions of the three types of dependence between criteria: correlation, substitutivity/complementarity, and preferential dependence.

A precise definition of a positive correlation between criteria *i* and *j* is defined by the following inequality:

$$v(ij) < v(i) + v(j). \tag{3}$$

More generally, if criteria *i* and *j* are positively correlated. then the marginal contribution of *j* to every combination $T \cup i$ of criteria that contains *i* is strictly less than the marginal contribution of *j* to the same combination when *i* is excluded, i.e.,

$$v(T \cup ij) - v(T \cup i) < v(T \cup j) - v(T) \text{ for } T \subseteq N \backslash ij \tag{4}$$

which expresses a negative interaction or a negative synergy between criteria *i* and *j*. Similarly, a negative correlation between criteria *i* and *j* is defined by the following inequality:

$$v(T \cup ij) - v(T \cup i) > v(T \cup j) - v(T) \text{ for } T \subseteq N \backslash ij. \tag{5}$$

The second type of dependence is that of substitutivity between criteria *i* and *j*. Two criteria, *i* and *j*, are almost substitutive or interchangeable, and so the importance of the pair $\{i, j\}$ is close to the importance of the single criterion, *i* and *j*, even in the presence of the other criteria. More precisely, we have

$$v(T) < \begin{Bmatrix} v(T \cup i) \\ v(T \cup j) \end{Bmatrix} \approx v(T \cup ij) \text{ for } T \subseteq N \backslash ij \tag{6}$$

where $\approx$ means approximately equal.

Finally, the preferential dependence and its opposite, the preferential independence, of the decision maker can be expressed by a weak order $\gtrsim$ (a strongly complete and transitive binary relation). This preference relation can be considered as a preference relation on $\mathbb{R}^n$.

Let e_i be the characteristic vector in $\{0,1\}^n$, i.e., a vector with zero components, except for a 1 in the *i*th component. Define the notation

$$xSy \equiv \sum_{i\in S} x_i e_i + \sum_{i\in N\setminus S} y_i e_i,\ x, y \in \mathbb{R}^n,\ S \subseteq N.$$

It follows that the subset S of criteria is said to be preference independent of $N\setminus S$ if for all $x, x', y, z \in \mathbb{R}^n$, we have

$$xSy \gtrsim x'Sy$$

If, and only if,

$$xSz \gtrsim x'Sz.$$

The preference of xSy over $x'Sy$ is not influenced by the common part, y.

This paper focuses on correlation, which is the most intuitive type of dependence. We will investigate the various properties of the weights, $v(S)$, $S \subseteq N$, of the coalition of criteria for the positive correlation among finite criteria set S. Those of the negative correlation can be derived similarly.

For the number of criteria $n = 3$, the weight variables are $v(1)$, $v(2)$, $v(3)$, $v(12)$, $v(13)$ and $v(23)$. The constraints of the positive correlation for an MADM problem with three criteria are the monotonicity condition of $v(S)$, the boundary conditions of $v(S)$ and the positive correlation conditions of $v(S)$ for $S \subseteq N = \{1,2,3\}$. In accordance with the monotonicity of $v(S)$, as stated earlier, it easily follows that

$$v(12) \geq v(1),\ v(12) \geq v(2)$$

$$v(13) \geq v(1),\ v(13) \geq v(3)$$

and

$$v(23) \geq v(2),\ v(23) \geq v(3).$$

The definition of the boundary conditions of $v(S)$ implies that

$$0 \leq v(1), v(2), v(3) \leq 1$$

and

$$0 \leq v(12), v(13), v(23) \leq 1.$$

From the positive correlation assumption (4), it follows that

$$v(12) - v(1) < v(2)$$

$$v(123) - v(13) < v(23) - v(3)$$

$$v(13) - v(1) < v(3)$$

$$v(123) - v(12) < v(23) - v(2)$$

$$v(23) - v(2) < v(3)$$

and

$$v(123) - v(12) < v(13) - v(1).$$

Therefore, the total number of the constraints of the positive correlation for an MADM problem with three criteria is 24. The variable number and the constraint number will increase if the number of criteria becomes high.

Consider the special case that

$$v(i_1, i_2, \ldots, i_k) = v(1,2, \ldots, k) + (i_1 + i_2 + \cdots + i_k - \frac{k(k+1)}{2})\alpha \text{ for } i_1, i_2, \ldots, i_k \in N \text{ and } k = 1,2, \ldots, n-1 \quad (7)$$

where $\alpha = v(2) - v(1)$. For an MADM problem with three criteria, it follows that the basic weight variables are $v(1)$, $v(2)$ and $v(12)$. We also have

$$v(3) = 2 \times v(2) - v(1)$$

$$v(13) = v(12) + v(2) - v(1)$$

and

$$v(23) = v(12) + 2 \times v(2) - 2 \times v(1).$$

Thus, the monotonicity condition of $v(S)$, the boundary conditions of $v(S)$ and the positive correlation conditions are respectively reduced to

$$v(12) \geq v(2)$$

$$v(12) + v(2) \geq 2 \times v(1),$$

$$0 \leq v(1) \leq 1$$

$$0 \leq 2 \times v(2) - v(1) \leq 1$$

$$v(12) + v(2) - v(1) \leq 1$$

$$v(12) + 2 \times v(2) - 2 \times v(1) \leq 1,$$

and

$$v(12) - v(1) < v(2)$$

$$0.5 + v(1) - v(2)/2 < v(12).$$

Therefore, the constraints of the positive correlation for an MADM problem with three criteria are

$$0 \leq v(1) \leq 1$$

$$0 \leq 2 \times v(2) - v(1) \leq 1$$

$$v(12) \geq v(2)$$

$$v(12) + v(2) \geq 2 \times v(1)$$

$$v(12) < v(1) + v(2)$$

$$0.5 + v(1) - v(2)/2 < v(12)$$

$$v(12) + v(2) - v(1) \leq 1$$

and

$$v(12) + 2 \times v(2) - 2 \times v(1) \leq 1.$$

The number of the less-than and equal-to constraints is eight, and those of the less-than constraints is two. The total number of the inequality constraints is ten. However, the region of the constraints is not a closed set. Following the data-envelopment analysis [22–24], we introduce the non-Archimedean number, ε. Using this small number, ε, we replace the less-than constraints

$$v(12) < v(1) + v(2)$$

and

$$0.5 + v(1) - v(2)/2 < v(12)$$

with

$$v(12) \leq v(1) + v(2) - \varepsilon$$

and

$$0.5 + v(1) - v(2)/2 + \varepsilon/2 \leq v(12),$$

respectively. The concise constraints of the feasible region of the positive correlation for an MADM problem with three criteria are described as follows:

$$0 \leq v(1) \leq 1$$

$$\frac{v(1)}{2} \leq v(2) \leq \frac{1+v(1)}{2}$$

and

$$\max\left\{v(2), 2 \times v(1) - v(2), 0.5 + v(1) - \frac{v(2)}{2} + \frac{\varepsilon}{2}\right\} \leq v(12)$$

$$\leq \min\{1 - v(2) + v(1), 1 - 2 \times v(2) + 2 \times v(1), v(1) + v(2) - \varepsilon\}$$

which is denoted by the polyhedral set, F_3.

For the number of criteria $n = 4$, the basic weight variables are $v(1)$, $v(2)$, $v(12)$ and $v(123)$. A similar argument shows that the constraints of the positive correlation for an MADM problem with four criteria are

$$0 \leq v(1) \leq 1$$

$$0 \leq 3 \times v(2) - 2 \times v(1) \leq 1$$

$$v(12) \geq v(2)$$

$$3 \times v(1) - 2 \times v(2) \leq v(12)$$

$$v(123) \geq v(12)$$

$$v(123) \geq 2 \times v(2) - v(1)$$

$$v(12) < v(1) + v(2)$$

$$v(123) < 2 \times v(12) - 2 \times v(1) + v(2)$$

$$0.5 + \frac{v(12)+v(1)-v(2)}{2} < v(123)$$

$$v(12) + 4 \times v(2) - 4 \times v(1) \leq 1$$

$$v(123) + v(2) - v(1) \leq 1$$

and

$$v(123) + 3 \times v(2) - 3 \times v(1) \leq 1.$$

The number of the less-than and equal-to constraints is eleven, and those of the less-than constraints is three. The total number of the inequality constraints is 14. Applying the non-Archimedean number, ε, the concise form of the feasible region of the positive correlation for an MADM problem with four criteria is described as follows:

$$0 \leq v(1) \leq 1$$

$$\frac{2 \times v(1)}{3} \leq v(2) \leq \frac{1 + 2 \times v(1)}{3}$$

$$\max\{v(2), 3 \times v(1) - 2 \times v(2)\} \leq v(12) \leq$$

$$\min\{1-4\times v(2)+4\times v(1), v(1)+v(2)-\varepsilon\}$$

and

$$\max\left\{v(12), 2\times v(1)-v(2), \frac{1+v(12)-v(2)+v(1)+\varepsilon}{2}\right\}$$
$$\le v(123) \le \min\{1-v(2)+v(1), 1-3\times v(2)+3\times v(1),$$
$$2\times v(12)-2\times v(1)+v(2)-\varepsilon\}$$

which is denoted by the polyhedral set, F_4.

Consider the constraints of the positive correlation for a general MADM problem with n $(n \ge 2)$ criteria. The basic weight variables are $v(1)$, $v(2)$, $v(12)$, …, and $v(1,2,\dots,n-1)$. The number of the less-than and equal-to constraints is $3n-1$ and those of the less than constraints is $n-1$. The total number of the inequality constraints is $4n-2$. Using the non-Archimedean number, ε, the concise feasible region of the positive correlation for a general MADM problem with n criteria is described as follows:

$$0 \le v(1) \le 1$$
$$\frac{(n-2)\times v(1)}{n-1} \le v(2) \le \frac{1+(n-2)\times v(1)}{n-2}$$
$$\max\{v(2), (n-1)\times v(1)-(n-2)\times v(2)\} \le v(12) \le$$
$$\min\{1-2(n-2)\times(v(2)-v(1)), v(1)+v(2)-\varepsilon\}$$
$$\dots \tag{8}$$

and

$$\max\left\{v(1,2,\dots,n-2), (n-2)\times v(2)-(n-3)\times v(1), \frac{1+v(1,2,\dots,n-2)-v(2)+v(1)+\varepsilon}{2}\right\}$$
$$\le v(1,2,\dots,n-1) \le \min\{1-v(2)+v(1), 1-(n-1)\times(v(2)-v(1)),$$
$$2\times v(1,2,\dots,n-2)-\mathrm{v}(1,2,\dots,n-3)+v(2)-v(1)-\varepsilon\}$$

which is denoted by the polyhedral set, F_n.

3. Pre-Emptive-Weights Goal-Programming

Consider the multicriteria programming problem of the positive correlation for an MADM problem with n criteria

$$\begin{gathered}\text{Max } v(1,2,\dots,n-1)\\ \text{Max } v(1,2,\dots,n-2)\\ \dots\\ \text{Max } v(1)\end{gathered} \tag{9}$$

Subject to $\left(v(1), v(2), v(12), \dots, v(1,2,\dots,n-1)\right) \in F_n$.

According to the boundary conditions of $v(S)$,

$$v(1,2,\dots,j) \le 1, \text{ for } j=1,2,\dots,n-1$$

it follows that the pre-emptive-weights goal-programming problem of the multicriteria programming (9) is

$$\text{Min } P_1d_1^- + P_2d_2^- + \cdots + P_{n-1}d_{n-1}^-$$

Subject to

$$v(1,2,\dots,n-1) + d_1^- - d_1^+ = 1$$

$$\begin{gathered} v(1,2,\dots,n-2)+d_2^- - d_2^+ = 1 \\ \dots \\ v(1)+d_{n-1}^- - d_{n-1}^+ = 1 \\ \big(v(1),v(2),v(12),\dots,v(1,2,\dots,n-1)\big) \in F_n. \end{gathered} \tag{10}$$

The symbols $P_1, P_2, \dots, P_{n-1}$ in the objective function stand for the pre-emptive weights determining the hierarchy of goals. Goals of higher priority levels are satisfied first, and only then may the lower priority goals be considered. The goals Max $v(1,2,\dots,j) \le 1$, for $j = 1,2,\dots,n-1$, cannot be achieved simultaneously. We concentrate on the deviation from the goals and try to find a solution which minimizes the deviations

$$v(1,2,\dots,j)+d_{n-j}^- - d_{n-j}^+ = 1, \text{ for } j = 1,2,\dots,n-1.$$

Variables d_i^- and d_i^+, $i = 1,2,\dots,n-1$, stand for the underachievement and overachievement, respectively. We do not care how big d_i^+ is, but we want d_i^- to be as small as possible.

The feasible region of the pre-emptive-weights goal-programming problem (10) is the bounded polyhedral set. The representation of a bounded polyhedral set can be determined by the extreme points. We adopt the extreme-point method to obtain the pre-emptive-weights goal-programming solutions.

For the number of criteria $n = 3$, the pre-emptive-weights goal-programming problem is described as follows:

$$\text{Min } P_1 d_1^- + P_2 d_2^-$$

Subject to

$$\begin{gathered} v(12)+d_1^- - d_1^+ = 1 \\ v(1)+d_2^- - d_2^+ = 1 \\ 0 \le v(1) \le 1 \\ \frac{v(1)}{2} \le v(2) \le \frac{1+v(1)}{2} \\ \max\left\{v(2), 2\times v(1)-v(2), 0.5+v(1)-\frac{v(2)}{2}+\frac{\varepsilon}{2}\right\} \le \\ v(12) \le \min\{1-v(2)+v(1), 1-2\times v(2)+2\times v(1), v(1)+v(2)-\varepsilon\}. \end{gathered}$$

The number of the basic weight variables is three. The number of the real constraints is ten. It follows that the number of the extreme points is $C_3^{10} = 120$, of which the feasible extreme points are

$$\big(v(1),v(2),v(12)\big) = \left(\tfrac{1}{2}+\tfrac{\varepsilon}{2}, \tfrac{1}{2}+\tfrac{\varepsilon}{2}, 1\right), \left(\tfrac{2}{3}, \tfrac{1}{3}+\varepsilon, 1\right)$$

and

$$(1-\varepsilon, 1-\varepsilon, 1).$$

This implies that the pre-emptive-weights goal-programming optimal solution is:

$$\big(v(1),v(2),v(12)\big) = (1-\varepsilon, 1-\varepsilon, 1).$$

The pre-emptive-weights goal-programming problem of the positive correlation for an MADM problem with four criteria is:

$$\text{Min } P_1 d_1^- + P_2 d_2^- + P_3 d_3^-$$

subject to

$$
\begin{gathered}
v(123)+d_1^- - d_1^+ = 1 \\
v(12)+d_2^- - d_2^+ = 1 \\
v(1)+d_3^- - d_3^+ = 1 \\
0 \le v(1) \le 1 \\
0 \le 3\times v(2) - 2\times v(1) \le 1 \\
v(12) \ge v(2) \\
3\times v(1) - 2\times v(2) \le v(12) \\
v(123) \ge v(12) \\
v(123) \ge 2\times v(2) - v(1) \\
v(12) \le v(1)+v(2)-\varepsilon \\
v(123) \le 2\times v(12) - 2\times v(1) + v(2) - \varepsilon \\
0.5 + \frac{v(12)+v(1)-v(2)+\varepsilon}{2} \le v(123) \\
v(12) + 4\times v(2) - 4\times v(1) \le 1 \\
v(123) + v(2) - v(1) \le 1 \\
v(123) + 3\times v(2) - 3\times v(1) \le 1.
\end{gathered} \tag{11}
$$

The number of the extreme points of the real constraints, F_4, is $C_4^{14} = 1001$. Thus, solving the extreme points is very difficult, and incurs a heavy computational burden. An efficient procedure is proposed and described as follows.

Since $v(123) \le 1$, the ideal solution is $v(123) = 1$. We substitute $v(123) = 1$ in F_4. That leads to

$$\max\left\{\frac{3\times v(2)-1}{2}, 0, v(2), 2v(2)-1\right\} \le v(1) \le \min\left\{\frac{3\times v(2)}{2}, 1\right\}$$

and

$$\max\left\{v(2), 3\times v(1) - 2\times v(2), \frac{1+\varepsilon-v(2)+2\times v(1)}{2}\right\} \le v(12)$$

$$\le \min\{1 - 4\times v(2) + 4\times v(1), v(1)+v(2)-\varepsilon, 1, 1 - v(1) + v(2) - \varepsilon\}.$$

To compare two inequalities head-to-head, the resulting constraints of F_4 are reduced to

$$v(2) \le v(1) \le \min\left\{\frac{3\times v(2)}{2}, 1\right\}$$

and

$$\max\left\{3\times v(1) - 2\times v(2), \frac{1+\varepsilon-v(2)+2\times v(1)}{2}\right\} \le v(12) \le$$

$$\min\{v(1)+v(2)-\varepsilon, 1, 1 - v(1) + v(2) - \varepsilon\}.$$

If $v(12) = 1$, then we have

$$1 \le 1 - v(1) + v(2) - \varepsilon$$

so

$$v(1) \le v(2) - \varepsilon$$

in contradiction to inequality $v(2) \le v(1)$. This implies that $v(12) < 1$. In accordance with the monotonicity of $v(S)$, it follows that

$$v(1) \le v(12) < 1.$$

Furthermore, if $v(1) = \frac{3\times v(2)}{2}$, then

$$v(2) = \frac{2\times v(1)}{3},$$

so

$$\max\{\tfrac{5\times v(1)}{3}, \tfrac{1+\varepsilon+4\times v(1)/3}{2}\} \le v(12) \le \min\{\tfrac{5\times v(1)}{3} - \varepsilon, 1, 1 - v(1)/3 - \varepsilon\}.$$

It follows that

$$\tfrac{5\times v(1)}{3} \le v(12) \le \tfrac{5\times v(1)}{3} - \varepsilon$$

and arrives at a contradiction.

Therefore, the reduced constraints of F_4 are

$$v(2) \le v(1)$$

and

$$\max\left\{3 \times v(1) - 2 \times v(2), \tfrac{1+\varepsilon-v(2)+2\times v(1)}{2}\right\} \le v(12) \le$$
$$\min\{v(1) + v(2) - \varepsilon, 1 - v(1) + v(2) - \varepsilon\}.$$

The inequality number in the above reduced constraints is much less than that in the constraints of F_4. Without doubt, solving the above reduced constraints is much easier than solving the constraints of F_4. The number of the possible extreme point is $C_3^5 = 10$. Solving the following systems of linear equations

$$\begin{cases} \frac{1+\varepsilon-v(2)+2\times v(1)}{2} = \mathrm{v}(12) \\ v(12) = v(1) + v(2) - \varepsilon \\ v(12) = 1 - v(1) + v(2) - \varepsilon \end{cases}, \begin{cases} \frac{1+\varepsilon-v(2)+2\times v(1)}{2} = \mathrm{v}(12) \\ v(12) = v(1) + v(2) - \varepsilon' \\ v(2) = v(1) \end{cases}$$
$$\begin{cases} \frac{1+\varepsilon-v(2)+2\times v(1)}{2} = \mathrm{v}(12) \\ v(12) = 1 - v(1) + v(2) - \varepsilon \\ v(2) = v(1) \end{cases}$$

and

$$\begin{cases} v(12) = v(1) + v(2) - \varepsilon \\ v(12) = 1 - v(1) + v(2) - \varepsilon \\ v(2) = v(1) \end{cases}$$

to obtain the feasible extreme points

$$\big(v(1), v(2), v(12), v(123)\big) = \left(\tfrac{1}{2}, \tfrac{1}{3} + \varepsilon, \tfrac{5}{6}, 1\right), \left(\tfrac{1}{3} + \varepsilon, \tfrac{1}{3} + \varepsilon, \tfrac{2}{3} + \varepsilon, 1\right),$$
$$(1 - 3\varepsilon, 1 - 3\varepsilon, 1 - \varepsilon, 1)$$

and

$$\left(\tfrac{1}{2}, \tfrac{1}{2}, 1 - \varepsilon, 1\right),$$

respectively, implies that the pre-emptive-weights goal-programming optimal solution for the goal-programming problem (11) is

$$(v(1), v(2), v(12), v(123)) = (1 - 3\varepsilon, 1 - 3\varepsilon, 1 - \varepsilon, 1).$$

For an MADM problem with $n = 5$ criteria, the number of the extreme points of the real constraints F_5 is $C_5^{18} = 8568$. A similar argument shows that the feasible extreme points are

$$\left(v(1), v(2), v(12), v(123), v(1234)\right) = \left(\frac{1}{2} - \varepsilon, \frac{1}{2} - \varepsilon, 1 - 3\varepsilon, 1 - \varepsilon, 1\right),$$
$$\left(\frac{1+2\varepsilon}{3}, \frac{1+2\varepsilon}{3}, \frac{2+\varepsilon}{3}, 1 - \varepsilon, 1\right), \left(\frac{1+6\varepsilon}{4}, \frac{1+6\varepsilon}{4}, \frac{1}{2} + 2\varepsilon, \frac{3+6\varepsilon}{4}, 1\right),$$

and

$$(1 - 6\varepsilon, 1 - 6\varepsilon, 1 - 3\varepsilon, 1 - \varepsilon, 1).$$

This implies that the pre-emptive-weights goal-programming optimal solution is

$$(v(1), v(2), v(12), v(123), v(1234)) = (1 - 6\varepsilon, 1 - 6\varepsilon, 1 - 3\varepsilon, 1 - \varepsilon, 1).$$

One can make several notable observations from the pre-emptive-weights goal-programming optimal solutions of the number of criteria $n = 3, 4, 5$. Firstly, we have $v(1,2, \dots, n-1) = 1$, which is an ideal solution, since $v(1,2, \dots, n-1) \leq 1$. We also derive $\max\{v(1), v(2)\} \leq v(12) \leq v(123) \leq \cdots \leq v(1,2, \dots, n-2) < 1$. Secondly, we obtain $v(1) = v(2)$ so $\alpha = 0$. Finally, the coefficients of $v(1,2, \dots, n-1)$, $v(1,2, \dots, n-2)$,…, $v(1)$ of ε are $0, -1, -3, \dots$, which is an arithmetic progression. These findings confirm the result of the pre-emptive-weights goal-programming problem for an MADM problem with positive correlation among n criteria, described as follows:

Theorem 1. *Consider the positive correlation for an MADM problem with n criteria. The number of the extreme points of the real constraints, F_n, is C_n^{4n-2} for $n = 2, 3, \dots$. The pre-emptive-weights goal-programming optimal solution is*

$$\left(v(1), v(2), v(12), \dots, v(1,2, \dots, n-2), v(1,2, \dots, n-1)\right) =$$
$$\left(1 - \frac{(n-1)(n-2)}{2}\varepsilon, 1 - \frac{(n-1)(n-2)}{2}\varepsilon, 1 - \frac{(n-2)(n-3)}{2}\varepsilon, \dots, 1 - \varepsilon, 1\right).$$

From the Theorem 1, the global-maximum value of $v(1,2, \dots, n-1)$ is equal to the ideal upper-bound of $v(1,2, \dots, n-1)$. However, the global-maximum value of $v(1,2, \dots, j)$, $j = 1, 2, \dots, n-2$, is far from its ideal upper-bound as the value of j decreases.

In the following, we use the example given by Marichal [9] to illustrate our results. Consider the problem of evaluating four students (A, B, C and D) with respect to three mathematical subjects (criteria): statistics (St), probability (Pr), and algebra (Al). These are given in in Table 1. The first two criteria, St and Pr, are correlated, since students good at statistics are also usually good at probability, and vice versa. Thus, these two criteria present some degree of redundancy. Four students have been evaluated as follows (marks are expressed on a scale from 0 to 20):

Table 1. Four students are evaluated on statistics, probability, and algebra.

Student	Statistics	Probability	Algebra
A	19	15	18
B	19	18	15
C	11	15	18
D	11	18	15

Assume that the first two subjects, St and Pr, are more important than the third, Al. In the case of the independent subjects/criteria, the weights could be 0.35, 0.35 and 0.3 for the statistics, probability, and algebra, respectively. From the weighted arithmetic means (1), we have $M(A) = 17.3$, $M(B) = 17.45$, $M(C) = 14.5$ and $M(D) = 14.65$. It follows that $B \succcurlyeq A \succcurlyeq D \succcurlyeq C$.

For the positively correlated subjects/criteria, we distinguish three types of $\alpha = v(2) - v(1)$: $\alpha < 0$, $\alpha = 0$, and $\alpha > 0$. Firstly, if $\alpha < 0$, we have $v(3) < v(2) < v(1)$ and $v(23) < v(13) < v(12)$. From the discrete Choquet integral (2), we obtain

$$\mathcal{C}_v(A) = 15\big(1 - v(13)\big) + 18\big(v(13) - v(1)\big) + 19v(1) = 15 + 3v(13) + v(1)$$

$$\mathcal{C}_v(B) = 15\big(1 - v(12)\big) + 18\big(v(12) - v(1)\big) + 19v(1) = 15 + 3v(12) + v(1)$$

$$\mathcal{C}_v(C) = 11\big(1 - v(23)\big) + 15\big(v(23) - v(3)\big) + 18v(3) = 11 + 4v(23) + 3v(3)$$

and

$$\mathcal{C}_v(D) = 11\big(1 - v(23)\big) + 15\big(v(23) - v(2)\big) + 18v(2) = 11 + 4v(23) + 3v(2).$$

for student *A*, *B*, *C* and *D*, respectively. This implies that $\mathcal{C}_v(B) > \mathcal{C}_v(A)$, $\mathcal{C}_v(D) > \mathcal{C}_v(C)$ and $\mathcal{C}_v(A) > \mathcal{C}_v(D)$, so that $B \succcurlyeq A \succcurlyeq D \succcurlyeq C$. The special case of F_3, as stated earlier, is the feasible extreme point $\big(v(1), v(2), v(12)\big) = \left(\frac{2}{3}, \frac{1}{3} + \varepsilon, 1\right)$.

In the case where $\alpha = 0$, we have $v(3) = v(2) = v(1)$ and $v(23) = v(13) = v(12)$. The discrete Choquet integrals (2) of the students *A*, *B*, *C* and *D* are $\mathcal{C}_v(A) = 15 + 3v(12) + v(1)$, $\mathcal{C}_v(B) = 15 + 3v(12) + v(1)$, $\mathcal{C}_v(C) = 11 + 4v(12) + 3v(1)$ and $\mathcal{C}_v(D) = 11 + 4v(12) + 3v(1)$, respectively. We derive $\mathcal{C}_v(A) = \mathcal{C}_v(B) > \mathcal{C}_v(C) = \mathcal{C}_v(D)$, so $A \approx B \succcurlyeq C \approx D$. Two special cases of F_3, as stated earlier, are the feasible extreme points $\big(v(1), v(2), v(12)\big) = \left(\frac{1}{2} + \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2}, 1\right)$ and $(1 - \varepsilon, 1 - \varepsilon, 1)$.

Thirdly, if $\alpha > 0$, we have $v(3) > v(2) > v(1)$ and $v(23) > v(13) > v(12)$. From the discrete Choquet integrals (2) of the students *A*, *B*, *C* and *D*, we have $\mathcal{C}_v(A) > \mathcal{C}_v(B)$, $\mathcal{C}_v(C) > \mathcal{C}_v(D)$ and $\mathcal{C}_v(B) > \mathcal{C}_v(C)$, so that $A \succcurlyeq B \succcurlyeq C \succcurlyeq D$.

We can see that the rankings of the students may be different with the change in the values of α. However, the decision-maker prefers students *A* and *B* to students *C* and *D*. The final decision is between student *A* or *B*.

4. Conclusions

For an MADM problem with positive correlation among n criteria, by introducing the non-Archimedean number, ε, the bounded polyhedral set, F_n, of the positively correlated weights, $v(S)$, related to the subset, S, of criteria, N, is presented. The basic weight variables are $v(1)$, $v(2)$, $v(12)$, …, and $v(1,2,\ldots,n-1)$. The number of the less-than and equal-to constraints is $4n - 2$. The number of the extreme points of F_n is C_n^{4n-2}. The preemptive-weights goal-programming optimal solution is $\big(v(1), v(2), v(12), \ldots, v(1,2,\ldots,n-2), v(1,2,\ldots,n-1)\big) = (1 - \frac{(n-1)(n-2)}{2}\varepsilon, 1 - \frac{(n-1)(n-2)}{2}\varepsilon, 1 - \frac{(n-2)(n-3)}{2}\varepsilon, \ldots, 1 - \varepsilon, 1)$. These theoretical global-maximum values of the positively correlated weights related to the subset of finite criteria can be useful for the practical applications of the positively correlated weights in an MADM problem.

Small and medium-sized enterprises (SMEs) are the main foundation for Taiwan's economic development and employment, but the enterprise's average life expectancy is only seven years. With the advent of the environmental, social, and corporate governance (ESG) era, we apply a two-additive Choquet integral that enables the modeling of various effects of importance and interactions among criteria. We use questionnaires and an expert/assessment committee to obtain the final consensus-weights of the importance and the possible interactions among criteria. The final consensus-weights of the importance and the possible interactions among criteria should be less than and equal to their maximum values derived from Theorem 1. In the future, we will analyze the characteristics of sustainable development of different industries and provide advice to SMEs.

Author Contributions: Both J.-H.C. and H.-C.T. contributed equally towards the accomplishment of this research. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Adeel, A.; Akram, M.; Koam, A.N.A. Group Decision-Making Based on *m*-Polar Fuzzy Linguistic TOPSIS Method. *Symmetry* **2019**, *11*, 735.
2. Fishburn, P.C. *Utility Theory for Decision Making*; Wiley: New York, NY, USA, 1970.
3. Jin, Y.; Kousar, Z.; Ullah, K.; Mahmood, T.; Pehlivan, N.Y.; Ali, Z. Approach to Multi-Attribute Decision-Making Methods for Performance Evaluation Process Using Interval-Valued T-Spherical Fuzzy Hamacher Aggregation Information. *Axioms* **2021**, *10*, 145.
4. Li, D.F. TOPSIS-based nonlinear-programming methodology for multiattribute decision making with interval-valued intuitionistic fuzzy sets. *IEEE Trans. Fuzzy Syst.* **2010**, *18*, 299–311.
5. Ozdemir, Y.S. A Spherical Fuzzy Multi-Criteria Decision-Making Model for Industry 4.0 Performance Measurement. *Axioms* **2022**, *11*, 325.
6. Ullah, K.; Hassan, N.; Mahmood, T.; Jan, N.; Hassan, M. Evaluation of Investment Policy Based on Multi-Attribute Decision-Making Using Interval Valued T-Spherical Fuzzy Aggregation Operators. *Symmetry* **2019**, *11*, 357.
7. Marichal, J.L. An axiomatic approach of the discrete Choquet integral as a tool to aggregate interacting criteria. *IEEE Trans. Fuzzy Syst.* **2000**, *8*, 800–807.
8. Sugeno, M. Theory of Fuzzy Integrals and Its Applications. Ph.D. Dissertation, Tokyo Inst. Technol., Tokyo, Japan, 1974.
9. Grabisch, M. The application of fuzzy integrals in multicriteria decisionmaking. *Eur. J. Operat. Res.*, **1996**, *89*, 445–456.
10. Wu, J.; Chen, F.; Nie, C.; Zhang, Q. Intuitionistic fuzzy-valued Choquet integral and its application in multicriteria decision making. *Inf. Sci.* **2013**, *222*, 509–527.
11. Grabisch, M. K-order additive discrete fuzzy measures and their representation. *Fuzzy Sets Syst.* **1997**, *92*, 167–189.
12. Büyüközkan, G.; Feyzioğlu, O.; Ersoy, M.S. Evaluation of 4PL operating models: A decision making approach based on 2-additive Choquet integral. *Int. J. Prod. Econ.* **2009**, *121*, 112–120.
13. Büyüközkan, G.; Ruan, D. Coquet integral based aggregation approach to software development risk assessment. *Inf. Sci.* **2010**, *180*, 441–451.
14. Xu, Z.S. Choquet integrals of weighted intuitionistic fuzzy information. *Inf. Sci.* **2010**, *180*, 726–736.
15. Tan, C.Q.; Chen, X.H. Intuitionistic fuzzy Choquet integral operator for multicriteria decision making. *Expert Syst. Appl.* **2010**, *37*, 149–157.
16. Xu, Z.S.; Xia, M.M. Induced generalized intuitionistic fuzzy operators. *Knowl. Based Syst.* **2011**, *24*, 197–209.
17. Baydaş, M.; Elma, O.E. An objectıve criteria proposal for the comparison of MCDM and weighting methods in financial performance measurement: An application in Borsa Istanbul. *Decision Making: Appl. Manag. Eng.* **2021**, *4*, 257–279. https://doi.org/10.31181/dmame210402257b.
18. Mukhametzyanov, I. Specific character of objective methods for determining weights of criteria in MCDM problems: Entropy, CRITIC and SD. *Decis. Mak. Appl. Manag. Eng.* **2021**, *4*, 76–105. https://doi.org/10.31181/dmame210402076i.
19. Murofushi, T.; Soneda, S. Techniques for reading fuzzy measures (III): Interaction index. In Proceedings of the 9th Fuzzy System Symposium, Sapporo, Japan, May 1993; pp. 693–696. (in Japanese)
20. Marichal, J.L.; Roubens, M. Determination of weights of interacting criteria from a reference set. *Eur. J. Operat. Res.* **2000**, *124*, 641–650.
21. Murofushi, T.; Sugeno, M. Non-additivity of fuzzy measures representing preferential dependence. In Proceedings of the 2nd International Conference on Fuzzy Logic & Neural Networks, Iizuka, Japan, 17–22 July 1992; pp. 617–620.
22. Charnes, A.; Cooper, W.W. The non-Archimedean CCR ratio for efficiency analysis: A rejoinder to Boyd and Färe. *Eur. J. Oper. Res.* **1984**, *15*, 333–334.
23. Charnes, A.; Cooper, W.W.; Rhodes, E. Measuring the efficiency of decision making units. *Eur. J. Oper. Res.* **1978**, *2*, 429–444.
24. Charnes, A.; Cooper, W.W.; Rhodes, E. Short communication: Measuring the efficiency of decision making units. *Eur. J. Oper. Res.* **1979**, *3*, 339.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.